MATH 158 BF

Algebra Review -- Basic Factoring

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This 1/2 unit module reviews prerequisite skills for factoring, then concentrates on the five factoring techniques typically included in an Elementary Algebra course.

To earn 1/2 unit of credit, a student must study the entire module and work all exercises. The last exercise set, 8c., features complete factoring. A student must submit to the math lab instructor, a notebook showing full details of having completed all the 8c exercises. Once the instructor verifies that all the 8c exercises have been worked correctly and with sufficient detail, a student will be given an examination on complete factoring.

If the examination is completed with suitable accuracy, credit will be awarded. Alternate examinations will be available if the accuracy of the first one is unsatisfactory.

C. A. Miller, January, 1987

MATH 158 bf

A. Objective

The objective of this module is to attain polynomial factoring skills at the level of elementary algebra.

B. Review of preliminary concepts

Polynomial factoring requires an understanding of several key words and some specific manipulative skills. A review of these words and skills preceeds the techniques for factoring polynomials. For a more complete treatment, see any one of the references listed on the last page.

1. Vocabulary associated with polynomials

a. A monomial is also called a simple term and generally has three ingredients. A numerical coefficient, a variable base(s) and the base's exponent. Coefficients and base(s) are joined by multiplication.

Examples:

i. $7x^3$ — exponent; coefficient; base

ii. $-225z^2$ — exponent; coefficient; base

iii. $11y$ — exponent, when unspecified, is understood to be 1; coefficient; base

iv. h^4 — exponent; when missing, coefficient is understood to be 1

v. 39

Note: It is o.k. to have only a coefficient. Both the base and its associated exponent are absent.

vi. $-36x^2y^5z$ — bases

Note: Each base has its own associated exponent.

b. Associated with each monomial is a property called its <u>degree</u>. Monomials consisting of non-zero coefficients only, have degree zero. For monomials with a single variable, its degree will simply be the exponent associated with the single variable. Although infrequently encountered in elementary algebra, the degree of a monomial with more than one base is found by adding the exponents of these bases. Hence for each mononial above, their degrees are:

i. 3

ii. 2

iii. 1

iv. 4

v. 0

vi. 8

c. A <u>polynomial</u> is either a monomial or the indicated sum or difference of several simple terms.

Examples:

i. $5x^3 - 17x$

ii. $x^2 - 3x - 28$

iii. $307a^4$

iv. $2x^4 - 13x^3 + 7x^2 - 8x + 15$

v. $3a^2b^7 - 17ab^4 + 11a^9b$

Even though "poly" means many, a monomial is considered to be a polynomial having exactly one term.

d. Each polynomial also has a <u>degree</u>. The polynomial's simple term which has highest degree is the degree of the polynomial. Hence, for each polynomial above the degree is

i. 3

ii. 2

iii. 4

iv. 4

v. 10

e. A polynomial having exactly <u>two</u> simple terms is called a <u>binomial</u>. Exactly <u>three</u> simple terms, <u>trinomial</u>.

2. Polynomial multiplication

Even the most complicated polynomial multiplication breaks down into several simple tasks of multiplying two monomials. Multiplying two monomials requires multiplying the numerial coefficient and applying a basic rule of exponents when each polonomial has the same base. Recall, that

$$x^a \cdot x^b = x^{a+b}$$

Exponent rule examples:

i. $x^{11} \cdot x^{13} = x^{24}$

ii. $y^4 \cdot y = y^5$

Multiplication of monomial examples:

i. $(7x^{11})(-8x^{13}) = -56x^{24}$

ii. $(-13y^4)(-2y) = 26y^5$

iii. $(2x^2y^3)(5x^4y)(-3x^5y^9) = -30x^{11}y^{13}$

Multiplication of polynomial examples:

i. (mono) (trinomial)

$(2x^3)(5x^8 - 11x^4 + x) = 10x^{11} - 22x^7 + 2x^4$

Note here that three mono x mono multiplications took place. That is, each simple term in the trinomial on the right was multiplied by $2x^3$.

ii. (binomial) (binomial)

$(2x + 3)(5x - 7) = 10x^2 + 15x - 14x - 21$

$+ 10x^2 + x - 21$

Note that with the horizontal alignment of binomials, the product can be found mentally by the FOIL method. That is

a. The first term of the product is obtained by multiplying the first terms ('F') of the binomials.

b. The middle term is obtained by adding the product of the two outermost terms, 'O', and the two innermost terms, 'I'.

c. The last term is obtained by multiplying the last terms, 'L', of the binomials,

iii. (binomial) (trinomial)

$$
\begin{array}{rrrrrr}
 & 5x^8 & -\ 11x^6 & + & x & \\
 & & 2x^4 & - & 3x^2 & \\
\hline
 & -15x^{10} & +\ 33x^8 & & -3x^3 & \\
10x^{12} & -22x^{10} & & +2x^5 & & \\
\hline
10x^{12} & -37x^{10} & +\ 33x^8 & +2x^5 & -3x^3 &
\end{array}
$$

Note here that "long multiplication" is more convenient, and that there were six mono x mono multiplications.

3. Comparing polynomial multiplication to polynomial addition

It is crucial to understand that with polynomials, multiplication is very different from addition. Recall that two monomials can be added only when they are <u>similar</u>. That is, they have the same base and associated exponents. Examples demonstrating <u>similar</u> and <u>not</u> <u>similar</u> follow.

<u>Similar</u>	<u>Not similar</u>
$-14x^2$ and $5x^2$	$-14x^2$ and $-14x^3$
$13y^{10}$ and $-y^{10}$	$7y^3$ and $4x^3$
$2a^2b^3c^8$ and $-7a^2b^3c^8$	$3a^2b^3$ and $5a^2c^3$

Using some of the above examples, both addition and multiplication are demonstrated.

$(-14x^2)+(5x^2) = -9x^2$	$(-14x^2)\cdot(5x^2) = -70x^4$
$(13y^{10}) + (-y^{10}) = 12y^{10}$	$(13y^{10})\ (-y^{10}) = -13y^{20}$
$7y^3 + 4x^3 = 7y^3 + 4x^3$	$(7y^3)\ (4x^3) = 28y^3x^3$

Note, not similar terms, and hence, cannot be added

4. Prime factorization

Another essential skill for polynomial factoring is prime factorization of integers. Each positive integers greater than one is either prime or composite. If composite, it can be factored as a product of primes. A list of primes less than 50 follows.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

All positive integers less than 50 not in the list are composite, that is, are composed of prime factors. The prime factorizations of some composite inte: gers are also shown. Use any one of the refererences on the last page to review the technique for prime factorization.

Prime factorization for some composite positive integers:

$$18 = 2 \cdot 3^2$$

$$20 = 2^2 \cdot 5$$

$$27 = 3^3$$

$$50 = 2 \cdot 5^2$$

$$325 = 5^2 \cdot 13$$

$$2907 = 3^2 \cdot 17 \cdot 19$$

5. Greatest Common Divisor for integers

Finding the Greatest Common Divisor, GCD, of two or more integers is essential for finding the GCD of two or more monomials. (Note that some texts use Greatest Common Factor instead of GCD when referring to monomials.) One can think of the GCD of several integers as the largest integer that evenly divides each of the several integers. For example, 12 is the largest integer that "goes into" 36, 156 and 240. That is, the GCD of 36, 156 and 240 is 12. Sometimes the GCD of several small integers is easy to find mentally. Clearly, the GCD of 30 and 42 is 6. For larger integers, the easiest technique for finding the GCD is

a. Write each integer in its prime factorization without using exponents.

b. Circle prime factors common to all lists.

c. Multiply the circled prime factors together to get the GCD. If there are no prime factors common to all lists, the GCD is 1.

Examples:

a. Find the GCD of 70 and 126

$$\begin{aligned} 70 &= 2\ 5\ 7 \\ 126 &= 2\ 3\ 3\ 7 \end{aligned} \qquad \text{the GCD} = 2 \cdot 7 = 14$$

b. Find the GCD of 36, 256 and 240

$$\begin{aligned} 36 &= 2\ 2\ 3\ 3 \\ 156 &= 2\ 2\ 3\ 13 \\ 240 &= 2\ 2\ 2\ 2\ 3\ 5 \end{aligned} \qquad \text{the GCD} = 2 \cdot 2 \cdot 3 = 12$$

c. Find the GCD of 48 and 875

$$\begin{aligned} 48 &= 2\ 2\ 2\ 2\ 3 \\ 875 &= 5\ 5\ 5\ 7 \end{aligned} \qquad \text{the GCD} = 1$$

6. Greatest Common Divisor for monomials

All we need now to extend the notion of GCD from integers to monomials is to include common factors of variables. The technique will be demonstrated by the use of examples.

a. Find the GCD of $70x^2$ and $126x^5$

$$\begin{aligned} 70x &= 2\ 5\ 7\ x\ x \\ 126x &= 2\ 3\ 3\ 7\ x\ x\ x\ x\ x \end{aligned} \qquad \text{the GCD is } 14\,x^2$$

b. Find the GCD of $18x^3y^4z^5$, $99xy^5$ and $117x^2y^3z^4$

$$\begin{aligned} 18x^3y^4z^5 &= 2\ 3\ 3\ x\ x\ x\ y\ y\ y\ y\ \ z\ z\ z\ z\ z \\ 99xy^5 &= 3\ 3\ 11\ x\ y\ y\ y\ y\ y \\ 117x^2y^3z^4 &= 3\ 3\ 13\ x\ x\ y\ y\ y\ z\ z\ z\ z \end{aligned}$$

The GCD is $3 \cdot 3 \cdot x \cdot y \cdot y \cdot y = 9xy^3$

Note that in order for a variable with an associated exponent to be included in the GCD, it must appear in <u>all</u> monomial factorizations. The smallest exponent for each variable that does appear in all prime factorization will then be its exponent in the GCD. Applying this technique to the examples above shortens the effort to find the GCD.

7. Squares and cubes of "small" integers

Finally, being able to quickly recognize perfect squares and cubes makes polynomial factoring easier. You are encouraged to create a list of the squares of integers from 1 to 20 and the cubes of integers from 1 to 10. Your format could be something like

integer	square	cube
1	1	1
2	4	8
3	9	27
4	16	64
.	.	.
.	.	.
.	.	.

C. Specific polynomial factoring techniques

1. Common monomial factoring

a. Method

i. Each term of the given polynomial is a monomial. Use the technique reviewed in B.6 for finding the GCD of these monomials.

ii. Divide each term of the given polynomial by this GCD. The quotient attained is the other factor.

iii. The answer is the indicated product of the GCD and the other factor.

b. Examples

1) Factor $30x + 42y$

step i. 6 is the GCD of $30x$ and $42y$

step ii. $\frac{30x + 42y}{6} = \frac{30x}{6} + \frac{42y}{6} = 5x + 7y$

step iii. Answer $6(5x + 7y)$

2) Factor $6x^4 + 5x^3 - 7x^2$

Step i. x^2 is the GCD of $6x^4 + 5x^3 - 7x^2$

Step ii. $\frac{6x^4 + 5x^3 - 7x^2}{x^2} = 6x^2 + 5x - 7$

Step iii. Answer $x^2(6x^2 + 5x - 7)$

3) Factor $28a^5b^2 - 35ab^4$

Step i. $7ab^2$ is the GCD of $28a^5b^2$ and $35ab^4$

Step ii. $\frac{28a^5b^2 + 35ab^4}{7ab^2} = \frac{28a^5b^2}{7ab^2} + \frac{35ab^4}{7ab^2} = 4a^4 + 5b^2$

Step iii. Answer: $7ab^2(4a^4 - 5b^2)$

4) Factor $3y^4 + y$

Step i. y is the GCD of $3y^4$ and y

Step ii. $\frac{3y^4 + y}{y} = \frac{3y^4}{y} + \frac{y}{y} = 3y^3 + 1$

Step iii. Answer: $y(3y^3 + 1)$

5) Factor $3y^4 + 7x^2$

Step i. 1 is the GCD of $3y^4 + 7x^2$

Step ii. Answer: $3y^4 + 7x^2$ is prime and hence cannot be factored. It is conventional to simply answer prime.

c. Exercises

Factor each of the polynomials when possible.

1) $11a + 11b$

2) $21 - 63c^2$

3) $12x - 54y^2$

4) $11x^3 + 15x^2$

5) $12q^2n^3p^4 - 5q^2n^4p$

6) $7a^2b^3c^5 - 11b^4c^2d + 13c^4d^2$

7) $3x^2 + 6xy - 9x^4$

8) $13x^3 + 14y^2 - 15z^4$

9) $39n^2q^4 - 65n^3q$

10) $51f^8g^5 + 119f^7g^8 - 17f^7g^5$

2. Difference of two squares

a. Method

i. Determine what the first term is the square of.

ii. Determine what the last term is the square of.

iii. The answer is the product of a sum and a difference. That is, the sum of the quantities identified in i) and ii) and their difference

b. Examples

1) Factor $49-b^2$

Step i) 49 is the square of 7

Step ii) b^2 is the square of b

Step iii) answer $(7+b)(7-b)$

2) Factor $4y^2-121$

Step i) $4y^2$ is the square of 2y

Step ii) 121 is the square of 11

Step iii) answer $(2y+11)(2y-11)$

3) Factor $9x^2-25z^2$

Step i) $9x^2$ is the square of 3x

Step ii) $25z^2$ is the square of 5z

Step iii) answer $(3x+5z)(3x-5z)$

4) Factor $64d^2+81$

answer prime because the <u>sum</u> of two squares will not factor

5) Factor $169p^6-225q^4t^{10}$

Step i) $169p^6$ is the square of $13p^3$

Step ii) $225q^4t^{10}$ is the square of $15q^2t^5$

Step iii) answer $(13p^3+15q^2t^5)(13p^3-15q^2t^5)$

c. Exercises. Factor each of the binomials when possible. Check answers by multiplication.

1) $n^2 - 100$

2) $289 - d^2$

3) $y^2 + 400$

4) $361x^2 - 1$

5) $9h^2 - 169v^2$

6) $64z^2 - 25$

7) $x^6 - 196c^{20}$

8) $a^2b^4c^6 - 25d^8$

9) $m^3 - 49x^2$

10) $121x^{30} - 2500y^6$

3. Combination of factoring techniques I: Common monomial and difference of two squares.

a. Method

i. Factor first into a product of the common monomial and the difference of two squares.

ii. Factor the difference of two squares.

iii. The answer is the product of the common monomial factor and the sum and difference factors found in ii.

b. Examples

1) Factor $99 - 11z^2$

Step i. $99 - 11z = 11(9 - z^2)$

Step ii. $9 - z^2 = (3 + z)(3 - z)$

Step iii. Answer $11(3 + z)(3-z)$

2) Factor $20xy^2 - 605xd^2$

Step i. $20xy - 605xd = 5x(4y^2 - 121d^2)$

Step ii. $4y^2 - 121d^2 = (2y + 11d)(2y - 11d)$

Step iii. answer $5x(2y + 11d)(2y - 11d)$

3) Factor $294a^6c^5 - 150b^{20}c^3$

Step i. $294a^6c^5 - 150b^{20}c^3 = 6c^3(49a^6c^2 - 25b^{20})$

Step ii. $49a^6c^2 - 25b^{20} = (7a^3c + 5b^{10})(7a^3c - 5b^{10})$

Step iii. answer $6c^3(7a^3c + 5b^{10})(7a^3c - 5b^{10})$

c. Exercises. Factor each of the binomials when possible. Check answers by multiplication.

1) $7x^2 - 28$

2) $169v^2s^3t^4 - 225s^3t^2$

3) $15zy^2 - 375z$

4) $21a^3b^2 + 189a^3d^4$

5) $64w^2e^6 - 484e^6q^2$

6) $51k^5j^2 - 17k^7$

7) $5x^7y^3 - 80x^3y^3$

8) $144n^7p - 4356n^5p^{11}$

4. Trinomial factoring of form $x^2 + bx + c$

a. Caution: A thorough understanding of multiplication of two binomials is essential. Practice binomial multiplication problems from any one of the references if needed.

b. Method

i. Begin by writing the answer as (x)(x)

ii. Find the factors of c whose sum is the coefficient of the middle term, b.

a) If c is positive, the desired factors of c must be alike in sign (both + or both -) and the desired sign will be the sign of b.

b) If c is negative, the desired factors of c must be unlike in sign (one + the other -) and the sign of b will be the sign of the factor of c with the larger absolute value.

iii. Use the FOIL method to check your factoring.

c) Examples

1) Factor $x^2 - 6x + 8$

Here $c = 8$, a positive number, so the factors of 8 must be alike in sign. Because $b = -6$, a negative number, both factors must be negative.

Possible factors of 8	Sum of these factors
(-1) (-8)	-9
(-2) (-4)	-6 ←

Hence, the answer is $(x-2)(x-4)$

2) Factor $y^2 - 5y - 24$

Here $c = -24$, a negative number, so the factors of -24 must be unlike in sign. Because $b = -5$, a negative number, the factor of -24 that possesses the larger absolute value is the negative factor.

Possible factors of -24	Sum of these factors
(1) (-24)	-23
(2) (-12)	-10
(3) (-8)	-5 ←
(4) (-6)	-2

Hence, the answer is $(y+3)(y-8)$

3) Factor $z^2 + 3z + 10$

Here $C=10$, a positive number, so the factors of 10 must be alike in sign. Because $b=3$, a positive number, both factors must be positive.

Positive factors of 10	Sum of the factors
(1) (10)	11
(2) (5)	7

Since neither of the sums of the factors is 3, the given trinomial is prime.

d) Exercises. Factor each of the trinomials when possible. Check answers by the FOIL method of multiplication.

1. $y^2 + 10y + 21$
2. $a^2 - 8a + 15$
3. $b^2 + 2b - 35$
4. $c^2 - 5c - 36$
5. $d^2 - 20d + 99$
6. $x^2 + 5x - 12$
7. $z^2 + 18z + 81$
8. $49 - 14g + g^2$
9. $h^2 - 14 - 5h$
10. $x^2 - 4x - 221$

5. Trinomial factoring of the form $ax^2 + bx + c$

a. Method

i) Find all possible pairs of binomials in which the product of the first terms is equal to ax^2 and the product of the second terms is equal to c.

a) If c is positive, the factors of c will be alike in sign (both + or both -) and the sign will be the sign of b.

b) If c is negative, the factors of c will be unlike in sign; one positive and one negative.

ii) Of these pairs, find the pair of binomials whose product yields the desired middle term; bx. If you exhaust all possible pairs of binomials without success, your trinomial is prime.

b. Examples

1) Factor $2x^2 + 19x + 35$

Factors of $2x^2$	Factors of 35	Possible factors of $2x^2 + 19x + 35$	Middle term of each product
(x) (2x)	(1) (35)	(x+1) (2x+35)	37x
	(5) (7)	(x+35) (2x+1)	71x
		(x+5) (2x+7)	17x
		(x+7) (2x+5)	19x ←

Hence, the answer is (x+7) (2x+5)

2) Factor $15x^2 - 7x - 4$

Both the possible first terms and the possible last terms can be in any order. Hence, there are many possible combinations, but either one or none of the possible combinations will produce the correct middle term, namely -7x. When you find the unique correct factors, stop. If no possibilities produce the correct middle term, the trinomial is prime. The best approach is to start right off by numerically walking through the possibilities on paper rather than trying to keep track of mental calculations.

Factors of $15x^2$	Factors of -4	Possible factos of $15x^2-7x-4$	Middle form of each product
(x) (15x)	(1) (-4)	(x+1) (15x -4)	11x
(3x) (5x)	(-1) (4)	(x-1) (15z+4)	-11x
	(2) (-2)	(x+4) (15z-1)	59x
		(x-4) (15x+f1)	-59x
		(x-2) (15x-2)	28x
		(x-2) (15x+2)	-28x
		(3x+1) (5x-4)	-7x ←
		(3x-4) (5x+1)	-17x
		(3x-1) (5x+4)	7x
		(3x+4) (5x-1)	17x
		(3x+2) (5x-2)	4x

Hence,the answer is (3x+1) (5x-4).

Note: All possible factors were shown for the sake of completeness. But once again, when you find the correct combination, stop.

3. Factor $14b + 8b^2 - 15$

First, rearrange the terms in descending powers of b to get $8b^2 + 14b - 15$. Next, your first terms will be either b and 8b or 2b and 4b. Your second terms will be either 1 and -15 or -1 and 15 or 3 and -5 or -3 and 5. After listing all possible factors and finding the middle term for each trial, you will find that the answer is (2b+5) (4b-3).

4. Factor $3x^2 + 6x - 10$

You are encouragaed to write down all possible factors and find the middle term of each possibility. You will discover that no possibility works! Hence $3x^2 + 6x - 10$ is prime.

c. Exercises. Factor each of the trinomials when possible. Check answers by the FOIL method of multiplication.

1. $3x^2 + 5x + 2$

2. $6y^2 - 13y - 5$

3. $35a^2 + 13a - 4$

4. $12c^2 - 13c - 14$

5. $25 + 4d^2 - 20d$

6. $5f + 7 - 2f^2$

7. $15d^2 + 47d - 78$

8. $56p^2 - 56 + 15p$

9. $9q^2 + 54q + 77$

10. $30 + 61r + 30r^2$

6. Combinations of factoring techniques II: Common monomial and trinomial factoring.

a) Method

i. Factor first into a product of the common monomial and the remaining trinomial.

ii. Factor the remaining trinomial using techniques shown in the last two sections.

iii. The answer is the product of the common monomial factor and the results of step ii).

b) Examples (details not shown)

1) Factor $40x^2 + 10x - 15$

$40x + 10x - 15 = 5\ (8x^2 + 2x - 3)$

but $8x^2 + 2x - 3 = (4x + 3)\ (2x - 1)$

hence, answer is $5(4x + 3)\ (2x - 1)$

2) Factor $65y^5 - 169y^4 + 78y^3$

$65y^5 - 169y^4 + 78y^3 = 13y^3(5y^2 - 13y + 6)$

but $5y^2 - 13y + 6 = (5y - 3)\ (y - 2)$

hence, answer is $13y^3(5y - 3)\ (y-2)$

3) Factor $301x^2yz^2 + 154x^2yz^3 + 105x^2yz$

$$301x^2yz^2 + 154x^2yz^3 + 105x^2yz = 7x^2yz(43z + 22z^2 + 15)$$
$$= 7x^2yz(22z^2 + 43z + 15$$
$$= 7x^2yz(11z + 5)\ (2z + 3)$$

c) Exercises. Factor each of the trinomials when possible. Check answers by multiplication.

1) $2d^2 - 32d + 120$

2) $3a^2c + 42ac + 147c$

3) $x^4y^5z^6 + 9x^4y^5z^5 + 14x^4y^5z^4$

4) $22x^2 - 22y^2 - 33xy$

5) $90n^2 - 9n - 189$

6) $20b^2c^2d - 64bc^2d + 12c^2d$

7) $56q^2 + 32q^3 - 60q$

8) $36x^2 + 18x + 108$

9) $-49a^2b^2 + 35a^3b - 42ab^3$

10) $208x^7 + 195x^6 + 52x^8$

7. Factoring the sum and difference of the two cubes.

a) In general, factoring can be thought of as "un multiplying". It is helpful to review special cases of multiplication to build confidence that corresponding special cases of factoring are valid. Consider $(A + B)\ (A^2 - AB + B^2)$ in vertical form

$$\begin{array}{rrrrl}
 & A^2 & - AB & + B^2 & \\
 & & A & + B & \\
\hline
 & A^2B & - AB^2 & + B^3 & \\
A^3 & - A^2B & + AB^2 & & \\
\hline
A^3 & + 0 & + 0 & + B^3 & = \quad A^3 + B^3
\end{array}$$

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Note that the product is the sum of two cubes. A similar special case of multiplication will lead to the difference of two cubes.

b) Method

Memorize the following:

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$C^3 - D^3 = (C - D)(C^2 + CD + D^2)$$

When factoring perfect cubes, it is essential to first recognize what one has the cubes of. That is, what expression plays the role of A, B, C or D.

c) Examples

1) Factor $x^3 + 64$
Here x is playing the role of A and 4 is playing the role of B. That is, $4^3 = 64$.
Hence, $(x^3 + 64) = (x + 4)(x^2 - 4x + 16)$.

2) Factor $27 - y^6$
Here 3 is playing the role of C and y^2 is playing the role of D. That is, $3^3 = 27$ and $(y^2)^3 = y^6$. Hence,
$27 - y^6 = 3^3 - (y^2)^3 = (3 - y^2)(9 + 3y^2 + y^4)$.

3) Factor $\frac{1}{8}z^3 - 343q^3$
Here $\frac{1}{2}z$ is playing the role of C and 7q is playing the role of D. Hence,
$\frac{1}{8}z^3 - 343q^3 = (\frac{1}{2}z - 7q)(\frac{1}{4}z + \frac{7}{2}zq + 49q^2)$.

d) Exercises. Factor each of the binomials when possible. Some of the later problems may contain common monomial factors.
Check answers by multiplication.

1. $j^3 - 125$
2. $k^3 + 1$
3. $8m^3 - 729$
4. $216 - x^3y^3$
5. $64z^3 + 27$
6. $343z^3 - n^3$
7. $64a^5 - 8a^2$
8. $729x^4y^4 + 343xy^7$
9. $36x^3 + 18y^3$
10. $a^{12}y^9 - 8b^6z^{15}$

8. Combination of factoring techniques III -- Complete factoring

a. Method

i. Always look for common monomial factors first.

ii. If the exercise is a binomial, determine if both terms are either perfect squares or perfect cubes and factor accordingly. Remember that only the difference of two squares is factorable.

iii. If the exercise is a trinomial, attempt to factor as shown in either 4 or 5.

iv. If the exercise contains more than three terms, factoring techniques other than common monomial are beyond the level of this module.

b. Examples

1) Factor $4x^5 + 12x^4 - 20x^3 + 28x^2$

$4x^2$ is common to all four terms. Dividing each term by $4x^2$ yields the other factor. Hence, answer is $4x^2(x^3 + 3x^2 - 5x + 7)$. No additional factoring technique need be investigated since the other factor contains four terms.

2) Factor $3d^7 - 48d^3$

This binomial appears to have neither perfect squares nor perfect cube terms. But by extracting the common monomial it becomes

$$3d^7 - 48d^3 = 3d^3(d^4 - 16).$$

Here d^4 and 16 are both squares, hence

$$= 3d^3(d^2 + 4)(d^2 - 4)$$

d and 4 are also squares, so further factoring yields the final answer, namely

$$3d^3(d^2 + 4)(d + 2)(d - 2).$$

3) Factor $5x^{10} + 5x$

Since $5x$ is common to both terms, $5x^{10} + 5x = 5x(x^9 + 1)$. But $x^9 + 1$ is the sum of two cubes with x^3 playing the role of A, and 1 playing the role of B.

Hence $5x(x^9+1) = 5x(x^3 + 1)(x^6 - x^3 + 1)$
$= 5x(x+1)(x^2 - x + 1)(x^6 - x^3 + 1)$.

The last two examples demonstrate the importance of asking yourself if further factoring is possible before considering the factoring process complete.

c. Exercises. Factor completely. Check answers by multiplication.

1) $7x^3y - 7xy^3$

2) $10b^2 + 280 - 110b$

3) $12c^2d^2 + 14d^2c - 40d^2$

4) $5a^4 - 80b^4$

5) $51x^4y^3 - 68x^3y^4 + 85x^2y^5 - 34x^5y^2$

6) $13g^4h^2 - 351gh^5$

7) $30x^2y + 20y + 70xy$

8) $21x^4 + 84z^4$

9) $2n^4 - 162q^4$

10) $11n^4 - 5632n$

11) $42t^2 + 252 + 231t$

12) $2t^2v^4 - 18t^4$

13) $15n^3 - 180nd^2 - 15n^2d$

14) $fg^2x - 2fgx + fx$

15) $8m^3n^6 - k^{12}$

16) $20x^5y^2 + 108x^2y^2$

17) $12w^2j + 36j^3 - 42wj^2$

18) $32n^8 - 2q^4$

19) $42a^5b^2 + 112a^4b^2 - 490a^3b^2$

20) $64x^{15}y^{6} - 125z^{12}t^{3}$

21) $15a^{6} - 240b^{4}$

22) $180d^{4} + 384d^{3} - 84d^{2}$

23) $4b^{11} + 4b^{5}c^{21}e^{9}$

24) $11x^{5}y^{5} + 66x^{4}y^{5} + 110x^{3}y^{5}$

25) $7h^{3}m^{3} + 175m^{7}h$

26) $432p^{7}q^{4} - 250pq^{25}$

27) $4r^{3}s^{3}t^{7} - 196r\,s^{5}t^{7}$

28) $45g^{5}h^{5} - 48g^{4}h^{6} - 84g^{3}h^{7}$

29) $21n^{7}p^{3} - 35n^{5}p^{5} + 49n^{4}p^{7} - 77n^{3}p^{3}$

30) $189q^{7}v - 56q^{3}v^{3}$

31) $546a^{5}b^{5} - 1960a^{4}b^{6} - 154a^{3}b^{7}$

32) $108c^{4}d^{2} - 1372cd^{11}e^{18}$

References

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